Displacement Current

Amper's law states: $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Integrate over a surface $\rightarrow$ $\underbrace{\int_S (\vec{\nabla} \times \vec{H}) \cdot \vec{ds}}_{\text{Stoke's Theorem}} = \int_S \vec{J} \cdot \vec{ds} + \int_S \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}$

D: electric flux density or also called electric displacement.

$$\boxed{\oint_C \vec{H} \cdot \vec{dl} = I_c + \underbrace{\int_S \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}}}$$ (Amper's law in integral form)

This term must also have same unit of amper. Therefore it is called displacement current, I_d.

Displacement current: $\boxed{I_d \triangleq \int_S \vec{J_d} \cdot \vec{ds} = \int_S \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}}$

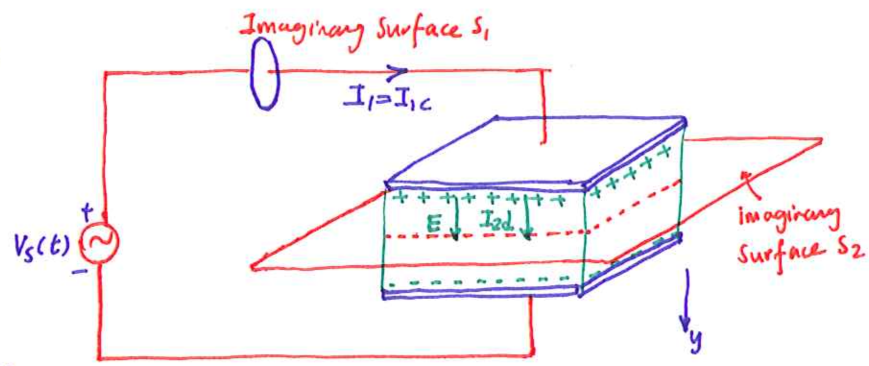
So the displacement current density is J_d: $J_d = \frac{\partial D}{\partial t}$, and the total current is the sum of the current I_c and the displacement current I_d =>

$$\oint \vec{H} \cdot \vec{dl} = I_c + I_d = I$$

A common example for I_d, is in a capacitor. Consider the circuit shown here with:

$V_s(t) = V_0 \cos \omega t$

Let's calculate I_d and I_c in the conductor (at surface S_1) and in the capacitor (at S_2).



In the conductor $D = E = 0$ $\rightarrow$ $J_{1d} = \frac{\partial D_1}{\partial t} = 0$ $\rightarrow$ $I_{1d} = 0$

To calculate I_{1c}, we use the circuit theory: $I_{1c} = C \frac{dV_c}{dt} = C \frac{d}{dt}(V_0 \cos \omega t) = -CV_0 \omega \sin \omega t$

So the total current through S_1 is: $I_1 = I_{1c} = -CV_0 \omega \sin \omega t$

As for the current through S_2, since the space between the conductor plates is filled with a dielectric, ε, it is insulator and no moving charge exists. So $I_{2c} = 0$

To find I_{2d}, we need to use $\vec{I}_d = \int_s \vec{J}_d \cdot \vec{ds} = \int \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}$:

we know that $\vec{E}$ is in y direction, and if the voltage across the plates is V_c, the magnitude of field is $E = \frac{V_c}{d}$ (becaus $V = \int E \cdot dl$). So we can write:

$$\vec{E} = \hat{y}\frac{V_c}{d} = \hat{y}\frac{V_0}{d}\cos\omega t \rightarrow \vec{D} = \varepsilon\vec{E} = \hat{y}\frac{\varepsilon V_0}{d}\cos\omega t$$

$$I_{2d} = \int_s \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds} = \int_A \left[\frac{\partial}{\partial t}\left(\hat{y}\frac{\varepsilon V_0}{d}\cos\omega t\right)\right] \cdot (\hat{y}\, ds) = -\frac{\varepsilon A}{d} V_0 \omega \sin\omega t$$

$$I_{2d} = -CV_0\omega\sin\omega t \qquad \left(\text{where we used } C = \varepsilon\frac{A}{d}\right)$$

It is seen that I_{2d} is equal to I_{1c} in the wire, which confirms continuity of current. It is important to notice that the displacement current doesn't carry real charge, but it still behaves like a real current.

Example Displacement current density:

A wire carrys current $I_c = 2\sin\omega t$ (mA) with $\omega = 10^9$ rad/s. If the wire has $\sigma = 2\times10^7$ S/m, and $\varepsilon_r = 1$, find the displacement current.

Solution: We have $I_c = JA = \sigma E A \rightarrow E = \frac{I_c}{\sigma A} = \frac{2\times10^{-3}\sin\omega t}{2\times10^7 A} = \frac{10^{-10}}{A}\sin\omega t \;(\frac{V}{m})$

$$D = \varepsilon E = \varepsilon_0\varepsilon_r E = \frac{\varepsilon_0 \times 10^{-10}}{A}\sin\omega t$$

$$I_d = J_d A = \frac{\partial D}{\partial t} A = \underset{\substack{\downarrow \\ 8.85\times10^{-12}}}{\varepsilon_0} \times 10^{-10}\,\omega\cos\omega t = 0.885\times10^{-12}\cos\omega t \;(A) = 0.885\times10^{-9}\cos\omega t \;(mA)$$

We notice that I_c and I_d have 90° phase difference. Also I_d is much smaller than I_c, which why we usually ignore it in good conductors.

Boundary Conditions for Electromagnetics

It turns out that same boundary conditions as we derived in the static form applies to the case of dynamic fields:

$$\hat{n}_2 \times (E_1 - E_2) = 0$$

$$\hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$

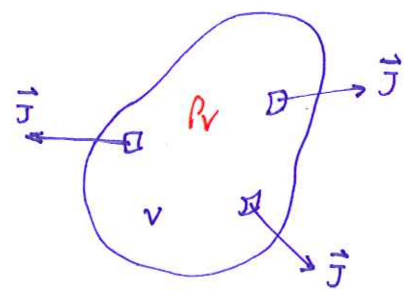
$$\hat{n}_2 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

$$\hat{n}_2 \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

Charge-Current Continuity Relation

Under static fields, there is no connection between ρ_v and $\vec{J}$. However, in the time-varying case there exists a connection between ρ_v and $\vec{J}$. Consider a volum of v surrounded by surface s.

The net current out from this surface is:



$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_v \rho_v \, dv$$

$$\downarrow$$

$$\oint_s \vec{J} \cdot \vec{ds} = -\frac{d}{dt}\int_v \rho_v \, dv$$

divergence theorem $\downarrow$

$$\int_v \vec{\nabla} \cdot \vec{J} \, dv = -\frac{d}{dt}\int_v \rho_v \, dv \quad \rightarrow \quad \boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}}$$

charge-current continuity relation or simply charge continuity equations.

This means that if the net charge in volume v is not changing with time (i.e. $\frac{\partial \rho_v}{\partial t} = 0$), the net current flowing out of ΔV is zero, or we can say the current flowing into the volume is equal to the total current flowing out from the surface: $\vec{\nabla} \cdot \vec{J} = 0$.

In integral form this gives:

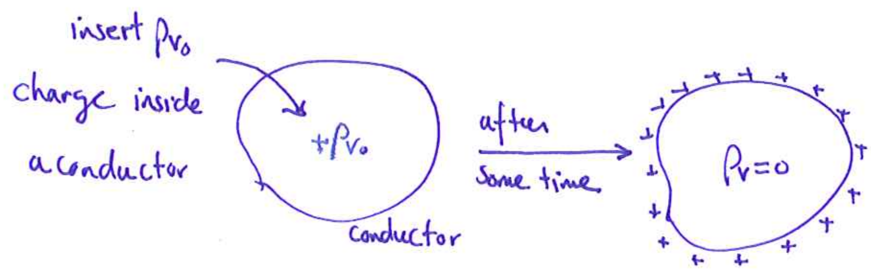
$$\boxed{\oint_s \vec{J} \cdot \vec{ds} = 0 \quad \text{(kirchhoff's current law)} \quad \left(\text{valid when } \frac{\partial \rho_v}{\partial t} = 0\right)}$$

$\Rightarrow \sum_i I_i = 0$ in circuit theory

Free-charge Dissipation in a Conductor

What happens if we insert some amount of charge ρ_{v_0} inside a conductor?

We know that the field inside a conductor is zero at equilibrium. In other word, the net charge inside a conductor is zero. Therefore the inserted excess charge with shift the electrons inside the conductor so that all the charges go to the surface of the conductor and inside the conductor we'll have $\rho_v = 0$:



How fast this charge transfer happens?

Let's use the continuity equation: $\vec{\nabla} \cdot \vec{J} = -\dfrac{\partial \rho_v}{\partial t}$

$$\vec{J} = \sigma \vec{E} \Rightarrow \nabla \cdot (\sigma \vec{E}) = \sigma \vec{\nabla} \cdot \vec{E} = -\frac{\partial \rho_v}{\partial t}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_v \Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\varepsilon} \Rightarrow \sigma \vec{\nabla} \cdot \vec{E} = \frac{\sigma}{\varepsilon} \rho_v = -\frac{\partial \rho_v}{\partial t} \rightarrow \frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0$$

$$\rightarrow \rho_v = \rho_{v_0} e^{-\frac{\sigma}{\varepsilon} t} \qquad \text{if we define } \tau_r \triangleq \frac{\varepsilon}{\sigma} \rightarrow \rho_v = \rho_{v_0} e^{-t/\tau_r} \quad \left(\frac{C}{m^3}\right)$$

$\tau_r = \frac{\varepsilon}{\sigma}$ is the relaxation time constant in the conductor. For example at $t = \tau_r$ the initial charge value of ρ_{v_0} becomes $\frac{\rho_{v_0}}{e} \simeq 0.37 \rho_{v_0}$ and at $t = 3\tau_r$ it becomes $\simeq 0.05 \rho_{v_0}$.

If we take $\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12} \frac{F}{m}$ and $\sigma = 5.8 \times 10^{7}$ S/m $\rightarrow \tau_r \simeq 1.53 \times 10^{-19}$ s. So the charge dissipation process in a conductor is extremely fast. On the other hand, for an insulator like mica that $\varepsilon = 6\varepsilon_0$ and $\sigma = 10^{-15}$ S/m, $\tau_r = 5.31 \times 10^{4}$ s or $\simeq 14.8$ hours!

Electromagnetic Potentials

For a time varying field we have: $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

since: $\vec{B} = \vec{\nabla} \times \vec{A} \rightarrow \vec{\nabla} \times \vec{E} = -\dfrac{\partial}{\partial t}(\nabla \times A)$

$$\rightarrow \nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0 \quad \text{if we define: } E' = E + \frac{\partial A}{\partial t} \rightarrow \nabla \times E' = 0$$

This means that E' is conservative. According to the rules of vector calculus, for a conservative field E', we can express it as the gradient of a scalar:

$$\vec{E}' = -\vec{\nabla} V \implies E + \frac{\partial A}{\partial t} = -\nabla V \implies$$

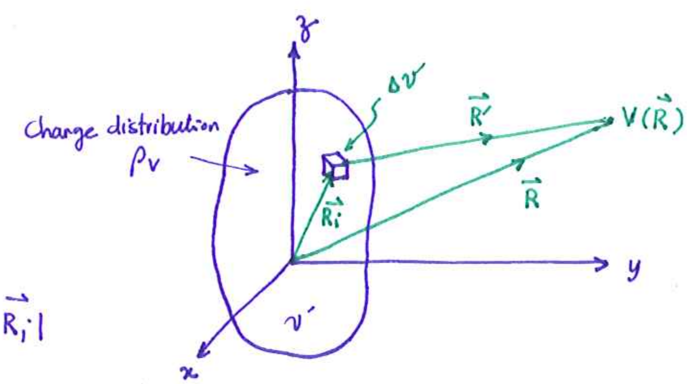
$$\boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}} \quad \text{(Dynamic)}$$

So when V and $\vec{A}$ are known, we can find $\vec{E}$ from the above relation and $\vec{B}$ from $\vec{B} = \vec{\nabla} \times \vec{A}$.

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \quad \text{(Dynamic)}$$

Retarded Potentials

Consider the charge distribution ρ_v shown in the picture. The medium is a perfect dielectric with pemitivity ε. We have for the potential at $\vec{R}$:



$$V(\vec{R}) = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_v(\vec{R}_i)}{R'} dv \quad \text{where } R' = |\vec{R} - \vec{R}_i|$$

if ρ_v is time varying, we may think that: $V(\vec{R}, t) = \dfrac{1}{4\pi\varepsilon} \displaystyle\int_{v'} \dfrac{\rho_v(\vec{R}_i, t)}{R'} dv'$

However, this is not exactly true as a time variation in ρ takes time to affect V at $\vec{R}$.

The time delay is equal to $t' = \dfrac{R'}{u_p}$, where u_p is the velocity of propagation in the medium between the charge and the observed point:

$$\boxed{V(\vec{R}, t) = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_v(\vec{R}_i, t - R'/u_p)}{R'} dv'} \quad \text{Retarded scalar potential}$$

If the medium is vacuum, $u_p = c$ velocity of light.

Similarly, the retarded vector potential $\vec{A}(\vec{R}, t)$ is:

$$\boxed{\vec{A}(\vec{R}, t) = \frac{\mu}{4\pi} \int_{v'} \frac{\vec{J}(\vec{R_i}, t - R'/u_p)}{R'} \, dv' \quad (\mathrm{Wb/m})}$$

Time-Harmonic Potentials

Since ρ_v and J are related with the continuity equation, A and V are connected, so are E and B. Also since E and B linearly depend on V and A, they all make a system that obeys linear rules and they all have same shape. So we can take advantage of sinusoidal-time functions to determine the response of the system due to a source with any type of time dependent by using fourier expansion.

We will now look at the equations for the case of steady state sine functions. We often call this time-harmonic responses. Suppose that $\rho_v(R_i, t)$ is a sinusoidal-time function:

$$\rho_v(\vec{R_i}, t) = \rho_v(\vec{R_i}) \cos \omega t$$

We will hence use the phasor method to simplify the equations:

$$\rho_v(R_i, t) = \mathrm{Re}\left[\underbrace{\tilde{\rho}_v(R_i)}_{\text{Phasor}} e^{j\omega t} \right]$$

Now we define the retarded charge density $\rho_v(\vec{R_i}, t - \frac{R'}{u_p})$ in phasor form by replacing t with $t - \frac{R'}{u_p}$:

$$\begin{aligned} \rho_v(\vec{R_i}, t - \frac{R'}{u_p}) &= \mathrm{Re}\left[\tilde{\rho}_v(\vec{R_i}) e^{j\omega(t - R'/u_p)} \right] \\ &= \mathrm{Re}\left[\tilde{\rho}_v(\vec{R_i}) e^{-j\omega R'/u_p} e^{j\omega t} \right] \\ &= \mathrm{Re}\left[\tilde{\rho}_v(\vec{R_i}) e^{-jkR'} e^{j\omega t} \right] \end{aligned}$$

Since $k = \frac{\omega}{u_p}$ =>

k is wavenumber or phase constant of the medium. We usually called it β but for lossless dielectric, it is often called k, wavenumber.

Similarly for potential:

$$\begin{aligned} V(\vec{R}, t) &= \mathrm{Re}\left[\tilde{V}(\vec{R}) e^{j\omega t} \right] \\ &= \mathrm{Re}\left[\frac{1}{4\pi\varepsilon} \int_{v'} \frac{\tilde{\rho}_v(\vec{R_i}) e^{-jkR'}}{R'} e^{j\omega t} \, dv' \right] \end{aligned}$$

$$\rightarrow \quad \boxed{\tilde{V}(R) = \frac{1}{4\pi\varepsilon} \int \frac{\tilde{\rho}_v(R_i)\, e^{-jkR'}}{R'}\, dv' \quad (V)}$$

Similarly for the vector potential $A(\vec{R},t) = \mathcal{Re}\left[\tilde{A}(\vec{R})\, e^{j\omega t}\right]$ with:

$$\boxed{\tilde{A}(\vec{R}) = \frac{\mu}{4\pi} \int_{v'} \frac{\tilde{J}(\vec{R}_i)\, e^{-jkR'}}{R'}\, dv'}$$

$\tilde{J}(\vec{R}_i)$ is the phasor for $\vec{J}(\vec{R}_i, t)$.

The magnetic field phasor $\tilde{H}$ corresponding to $\tilde{A}$ is from $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow$

$$\vec{\tilde{H}} = \frac{1}{\mu} \vec{\nabla} \times \vec{\tilde{A}}$$

Since the differentation in time domain is equivalent to multiplication by $j\omega$ in the phasor domain, in a nonconducting medium $(\vec{J}=0)$, Ampére's law becomes:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial D}{\partial t} \xrightarrow[\vec{D} = \varepsilon \vec{E}]{\text{if } \vec{J}=0} \boxed{\nabla \times \vec{\tilde{H}} = j\omega\varepsilon \vec{\tilde{E}} \quad \text{or} \quad \vec{\tilde{E}} = \frac{1}{j\omega\varepsilon} \vec{\nabla} \times \vec{\tilde{H}}} \quad \text{when } \vec{J}=0$$

The phasor vectors $\tilde{E}$ and $\tilde{H}$ also are related by the phasor form of Faraday's law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \boxed{\vec{\nabla} \times \vec{\tilde{E}} = -j\omega\mu \vec{\tilde{H}} \quad \text{or} \quad \vec{\tilde{H}} = -\frac{1}{j\omega\mu} \nabla \times \vec{\tilde{E}}}$$

Example

In a non-conducting medium with $\varepsilon = 16\varepsilon_0$ and $\mu = \mu_0$, the electric field intensity of an electromagnetic wave is given by: $\vec{E}(z,t) = \hat{x}\, 10 \sin(10^{10} t - kz)$ (V/m)

determine the associated magnetic field intensity $\vec{H}$ and find the value of k.

Solution: $E(z,t) = \hat{x}\, 10 \sin(10^{10} t - kz) \rightarrow \tilde{E}(z) = \hat{x}\, 10\, e^{-jkz}\, e^{-j\pi/2} = -\hat{x}\, j\, 10\, e^{-jkz}$

For $\tilde{H}$, from Faraday's law we have:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} \rightarrow \vec{\nabla}\times\tilde{E} = -j\omega\mu\tilde{H}$$

$$\rightarrow \vec{\tilde{H}} = -\frac{1}{j\omega\mu}\vec{\nabla}\times\vec{E} = -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -j10e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$\vec{\tilde{H}} = -\frac{1}{j\omega\mu}\left[\hat{y}\frac{\partial}{\partial z}\left(-j10e^{-jkz}\right)\right] = -\hat{y}\,j\frac{10k}{\omega\mu}e^{-jkz}$$

We have found $\tilde{H}$, but k is still unknown. to find k, we will calculate $\vec{E}$ from $\vec{H}$ using Ampere's law and compare with the given $\vec{E}$ to find k:

Amperes law: $\vec{\nabla}\times\vec{H} = \underbrace{\vec{J}}_{=0} + \frac{\partial D}{\partial t} \rightarrow \vec{\nabla}\times\vec{\tilde{H}} = j\omega\varepsilon\vec{\tilde{E}}$

$$\rightarrow \vec{\tilde{E}} = \frac{1}{j\omega\varepsilon}\vec{\nabla}\times\vec{\tilde{H}} = \frac{1}{j\omega\varepsilon}\left[-\hat{x}\frac{\partial}{\partial z}\left(-j\frac{10k}{\omega\mu}e^{-jkz}\right)\right]$$

$$= -\hat{x}\,j\frac{10k^2}{\omega^2\mu\varepsilon}e^{-jkz}$$

Comparing the two relations for $\vec{\tilde{E}} \Rightarrow \frac{k^2}{\omega^2\mu\varepsilon} = 1 \rightarrow k = \omega\sqrt{\mu\varepsilon} = 4\omega\sqrt{\mu_0\varepsilon_0}$

$$k = \frac{4\omega}{c} = \frac{4\times10^{10}}{3\times10^8} = 133 \text{ (rad/m)}$$

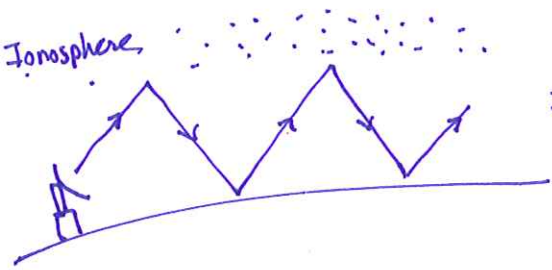
So we can write: $\vec{\tilde{H}} = -\hat{y}\,j\frac{10k}{\omega\mu}e^{-jkz} = \hat{y}\frac{10k}{\omega\mu}e^{-jkz}e^{-j\pi/2}$

$$\rightarrow \vec{H} = \hat{y}\frac{10k}{\omega\mu}\cos\left(\omega t - kz - \frac{\pi}{2}\right)$$

$$\vec{H} = \hat{y}\frac{10k}{\omega\mu}\sin(\omega t - kz) = \hat{y}\,0.11\sin(10^{10}t - 133z)$$

Unbounded EM Waves

When the EM wave is travelled in a material such as a transmission line, we call it EM traveling in a guided medium. It doesn't have to be a solid material. for example Earth's surface and ionosphere make parallel boundaries of a natural guiding structure for the propagation of short-wave radio transmissions in the HF band (3 – 30 MHz).

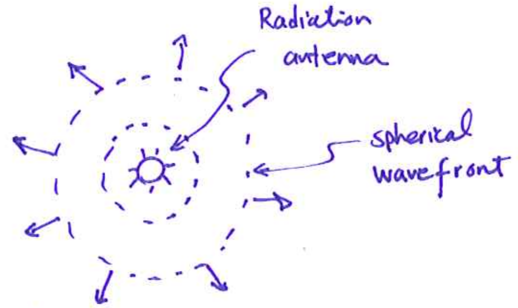


Ionosphere reflects back the wave to the Earth's surface.

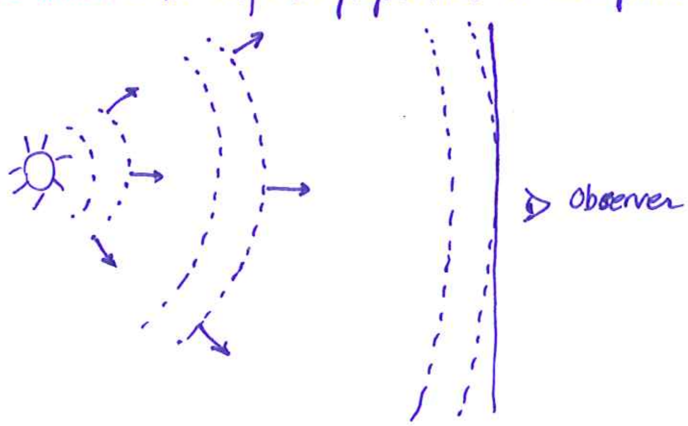
EM waves like light can also travel in unbounded media. Radio transmissions by antennas or the sun light are examples. In this chapter we focus on unbounded media.

Spherical wave:

Energy is emitted with a spherical wavefront from a source.



In far distance the wavefront appears approximately planar as if it is part of a uniform plane wave with uniform properties at all points.



In this chapter we focus on uniform plane wave in lossless and lossy media and in chap 8 will see how waves (planar and spherical) are reflected & transmitted through boundaries between dissimilar media.

Time-Harmonic Fields

If the time variation is sinusoidal in E, D, B, and H, we can use phasor form of Maxwell's equations:

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{\tilde{E}} = \tilde{\rho}_v/\varepsilon & (\vec{D}=\varepsilon\vec{E},\ \vec{\nabla}\cdot\vec{D}=\rho_v) \\ \vec{\nabla}\times\vec{\tilde{E}} = -j\omega\mu\vec{\tilde{H}} & (\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}) \\ \vec{\nabla}\cdot\vec{\tilde{H}} = 0 & (\vec{B}=\mu\vec{H},\ \vec{\nabla}\cdot\vec{B}=0) \\ \vec{\nabla}\times\vec{\tilde{H}} = \vec{\tilde{J}} + j\omega\varepsilon\vec{\tilde{E}} & (\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}) \end{array}}$$

Complex Permitivity

From Amperés law we have: $\underline{\vec{\nabla}\times\vec{\tilde{H}} = \vec{\tilde{J}} + j\omega\varepsilon\vec{\tilde{E}}}$

Since $\vec{J}=\sigma\vec{E} \rightarrow \vec{\nabla}\times\vec{\tilde{H}} = \sigma\vec{\tilde{E}} + j\omega\varepsilon\vec{\tilde{E}} = j\omega\underbrace{(\varepsilon - j\frac{\sigma}{\omega})}_{\varepsilon_c}\vec{\tilde{E}}$

we can define a complex permitivity as:

$$\boxed{\varepsilon_c \triangleq \varepsilon - j\frac{\sigma}{\omega}}$$

And rewrite the Amper's law as: $\vec{\nabla}\times\vec{\tilde{H}} = j\omega\varepsilon_c\vec{\tilde{E}}$

$$\varepsilon_c \triangleq \varepsilon - j\frac{\sigma}{\omega} \triangleq \varepsilon' - j\varepsilon'' \rightarrow \begin{array}{ll} \varepsilon' = \varepsilon & \text{Real part of } \varepsilon_c \\ \varepsilon'' = \frac{\sigma}{\omega} & \text{Imaginary part of } \varepsilon_c \end{array}$$

For a lossless medium with $\sigma=0 \Rightarrow \varepsilon''=0$ and $\varepsilon_c=\varepsilon$.

Wave Propagation for a Charge-free Medium

If there is no excess charge in the medium, $\rho_v=0$ and we have:

$$\boxed{\begin{array}{l} \vec{\nabla}\cdot\vec{\tilde{E}} = 0 \\ \vec{\nabla}\times\vec{\tilde{E}} = -j\omega\mu\vec{\tilde{H}} \\ \vec{\nabla}\cdot\vec{\tilde{H}} = 0 \\ \vec{\nabla}\times\vec{\tilde{H}} = j\omega\varepsilon_c\vec{\tilde{E}} \end{array}}$$

To describe the propagation of an EM wave, we need to derive the wave equation for $\vec{\tilde{E}}$ and $\vec{\tilde{H}}$ and solve them.

Let's start from: $\vec{\nabla} \times \vec{\tilde{E}} = -j\omega\mu \vec{\tilde{H}}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\tilde{E}}) = -j\omega\mu (\vec{\nabla} \times \vec{\tilde{H}})$$

$$= -j\omega\mu (j\omega\varepsilon_c \vec{\tilde{E}})$$

$$= \omega^2 \mu \varepsilon_c \vec{\tilde{E}}$$

we also know from chap. 3 that curl of the curl of $\vec{E}$ is given by:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\tilde{E}}) = \vec{\nabla} (\underbrace{\vec{\nabla} \cdot \vec{\tilde{E}}}_{=0 \text{ (no charge)}}) - \nabla^2 \vec{\tilde{E}}$$

where $\nabla^2 \vec{\tilde{E}}$ is laplacian given by: $\nabla^2 \vec{\tilde{E}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{\tilde{E}}$

$$\Rightarrow \quad -\nabla^2 \vec{\tilde{E}} = \omega^2 \mu \varepsilon_c \vec{\tilde{E}} \quad \text{or} \quad \nabla^2 \vec{\tilde{E}} + \omega^2 \mu \varepsilon_c \vec{\tilde{E}} = 0$$

This equation is called the homogeneous wave equation for $\tilde{E}$.

we define the propagation constant γ as:

$$\boxed{\gamma^2 \triangleq -\omega^2 \mu \varepsilon_c}$$

So we have: $\boxed{\nabla^2 \vec{\tilde{E}} - \gamma^2 \vec{\tilde{E}} = 0}$ wave equation in a charge-free medium

We may similarly derive: $\boxed{\nabla^2 \vec{\tilde{H}} - \gamma^2 \vec{\tilde{H}} = 0}$

The two equations look the same. So the solutions have similar form too.

Plane-Wave Propagation in Lossless Media

If the medium is nonconducting ($\sigma = 0$), the wave does not suffer any attenuation as it travels through the medium $\rightarrow$ lossless.

$$\sigma = 0 \Rightarrow \varepsilon_c = \varepsilon \Rightarrow \gamma^2 = -\omega^2 \mu \varepsilon$$

for lossless medium, we also define the wavenumber k as:

$$\boxed{k = \omega\sqrt{\mu\varepsilon}}$$

$$\Rightarrow \quad \nabla^2 \vec{\tilde{E}} + \omega^2 \mu \varepsilon_c \tilde{E} = 0 \quad \longrightarrow \quad \nabla^2 \vec{\tilde{E}} + k^2 \vec{\tilde{E}} = 0$$

Uniform Plane Wave

In cartesian coordinate, for $\vec{E}$ we have: $\vec{\tilde{E}} = \hat{x}\,\tilde{E}_x + \hat{y}\,\tilde{E}_y + \hat{z}\,\tilde{E}_z$

From the wave equation of $\nabla^2 \vec{\tilde{E}} + k^2 \vec{\tilde{E}} = 0 \Rightarrow$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\hat{x}\tilde{E}_x + \hat{y}\tilde{E}_y + \hat{z}\tilde{E}_z) + k^2(\hat{x}\tilde{E}_x + \hat{y}\tilde{E}_y + \hat{z}\tilde{E}_z) = 0$$

$\Rightarrow$ Each vector component on the left-hand side must be equal to zero:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{E}_x = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{E}_y = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{E}_z = 0$$

Since we are looking at uniform plane wave, if the plane is x-y plane, $\tilde{E}$ and $\tilde{H}$ are uniform on xy plane and: $\frac{\partial \tilde{E}_x}{\partial x} = \frac{\partial \tilde{E}_x}{\partial y} = 0$ and so on $\Rightarrow$

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} + k^2 \tilde{E}_x = 0$$

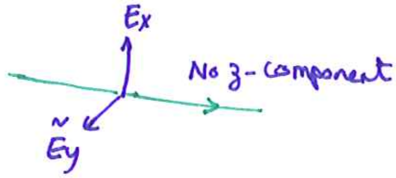
Similar expression holds for $\tilde{E}_y$, $\tilde{H}_x$ and $\tilde{H}_y$:

$$\frac{d^2\tilde{E}_y}{dz^2} + k^2\tilde{E}_y = 0 \qquad \frac{d^2\tilde{H}_x}{dz^2} + k^2\tilde{H}_x = 0 \qquad \frac{d^2\tilde{H}_y}{dz^2} + k^2\tilde{H}_y = 0$$

The remaining components of $\tilde{E}_z = \tilde{H}_z = 0$ are zero. To show this consider $\vec{\nabla} \times \vec{\tilde{H}} = j\omega\varepsilon_c \vec{\tilde{E}}$

The z component is: $\hat{z}\left(\underbrace{\frac{\partial \tilde{H}_y}{\partial x}}_{=0} - \underbrace{\frac{\partial \tilde{H}_x}{\partial y}}_{=0}\right) = \hat{z}\, j\omega\varepsilon \tilde{E}_z \implies \tilde{E}_z = 0$ Similarly $\tilde{H}_z = 0$.

This means that a plane wave has no electric or magnetic field component along its direction of propagation.



Let's solve the wave equation for $\tilde{E}_x$:

+z-direction → ← −z direction

$$\frac{d^2\tilde{E}_x}{dz^2} + k^2\tilde{E}_x = 0 \rightarrow \tilde{E}_x(z) = \tilde{E}_x^+(z) + \tilde{E}_x^-(z) = E_{x0}^+ e^{-jkz} + E_{x0}^- e^{jkz}$$

E_{x0}^- and E_{x0}^+ are constants to be determined from boundary conditions.

Let's assume for a case that $\tilde{E}_y = 0$ and the wave is travelling only in $+z$ direction:

$$\tilde{E}(z) = \hat{x}\,\tilde{E}_x^+(z) = \hat{x}\,E_{x0}^+ e^{-jkz}$$

To get $\tilde{H}$, use: $\vec{\nabla} \times \vec{\tilde{E}} = -j\omega\mu\tilde{H}$

$$\vec{\nabla} \times \vec{\tilde{E}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu(\hat{x}\tilde{H}_x + \hat{y}\tilde{H}_y + \hat{z}\tilde{H}_z)$$

$$\hat{x}(0) + \hat{y}\frac{\partial \tilde{E}_x^+(z)}{\partial z} - \hat{z}\underbrace{\frac{\partial \tilde{E}_x(z)}{\partial y}}_{=0} = -j\omega\mu(\hat{x}\tilde{H}_x + \hat{y}\tilde{H}_y + \hat{z}\tilde{H}_z)$$

$$\Rightarrow \quad \tilde{H}_x = 0$$

$$\begin{cases} \tilde{H}_y = -\dfrac{1}{j\omega\mu} \dfrac{\partial \tilde{E}_x^+(z)}{\partial z} \\ \tilde{H}_z = \dfrac{-1}{j\omega\mu} \underbrace{\dfrac{\partial E_x^+(z)}{\partial y}}_{=0} = 0 \end{cases} \quad \Rightarrow \quad \tilde{H}_y(z) = -\frac{1}{j\omega\mu} \frac{\partial}{\partial z} \left(\tilde{E}_{x0}^+ e^{-jkz} \right)$$

$$= \underbrace{\frac{k}{\omega\mu} \tilde{E}_{x0}^+}\, e^{-jkz} = \underbrace{H_{y0}^+}\, e^{-jkz}$$

$$\rightarrow \quad H_{y0}^+ = \frac{k}{\omega\mu} E_{x0}^+ \quad \longrightarrow \quad \frac{E_{x0}^+}{H_{y0}^+} = \frac{\omega\mu}{k}$$

The intrinsic impedance of a medium is the ratio of electric to magnetic field.

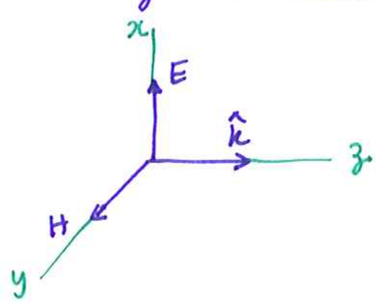
For the lossless media it is given by:

$$\boxed{\eta \triangleq \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}} \quad (\Omega)$$

We can summarize the results as:

$$\tilde{E}(z) = \hat{x}\, \tilde{E}_x^+(z) = \hat{x}\, E_{x0}^+ e^{-jkz}$$

$$\tilde{H}(z) = \hat{y}\, \frac{\tilde{E}_x^+(z)}{\eta} = \hat{y}\, \frac{E_{x0}^+}{\eta} e^{-jkz}$$



E and H are perpendicular to each other and both are perpendicular to the direction of wave travel. These properties characterize a transverse electromagnetic (TEM) wave.

Other examples of TEM waves are the spherical wave from an antenna or the EM wave in a coaxial cable ($\vec{E}$ is in $\hat{r}$ and $\vec{H}$ in $\hat{\varphi}$ direction)

In general E_{x0}^+ and H_{y0}^+ are complex functions. So they can have amplitudes of $|E_{x0}^+|$ and $|H_{y0}^+|$ and phases. For example if $E_{x0}^+ = |E_{x0}^+| e^{j\phi^+} \Longrightarrow$

$$\vec{E}(z,t) = \hat{x}\, |E_{x0}^+| \cos(\omega t - kz + \phi^+) \quad \text{and}$$

$$\vec{H}(z,t) = \hat{y}\, \frac{|E_{x0}^+|}{\eta} \cos(\omega t - kz + \phi^+)$$

E and H are in-phase. This is true for lossless medium.

Phase velocity: $u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$ (m/s)

wavelength: $\lambda = \frac{2\pi}{k} = \frac{u_p}{f}$ (m)

If the medium is vacuum: $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ and we have:

$$u_p = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times10^8 \text{ (m/s)}$$

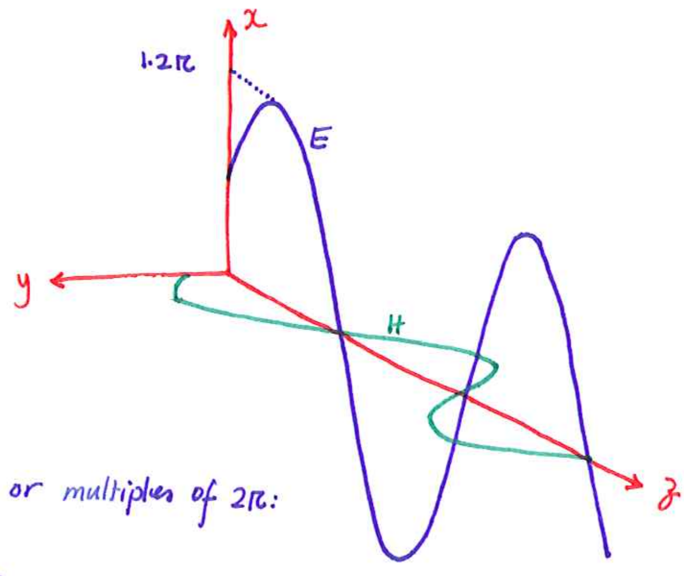
$$\eta = \eta_0 \triangleq \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,(\Omega) \simeq 120\pi \;(\Omega)$$

where c is the speed of light. η_0 is the intrinsic impedance of free space.

Example

The Electric field is traveling in +z direction in air along the x-axis. f is 1 MHz and the peak value of E is 1.2π (mV/m) and E is maximum at t=0 and z= 50m. Obtain expression for $\vec{E}(z,t)$ and $\vec{H}(z,t)$ and plot them at t=0 as a function of z.

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{1\times10^6} = 300\text{ m}$$



$$\rightarrow k = \frac{2\pi}{\lambda} = \frac{2\pi}{300}$$

$$\vec{E}(z,t) = \hat{x}\,|E^+_{x0}|\cos(\omega t - kz + \phi^+)$$

$$\vec{E}(z,t) = \hat{x}\,1.2\pi\cos(2\pi\times10^6 t - \frac{2\pi}{300}z + \phi^+)$$

$\vec{E}$ is max when the argument of cosine is zero or multiples of 2π:

$t=0$, $z=50$ $\Rightarrow$ $-\frac{2\pi}{300}z + \phi^+ = 0 \quad \rightarrow \phi^+ = \frac{2\pi\times50}{300} = \frac{\pi}{3}$

$$\Rightarrow \vec{E}(z,t) = \hat{x}\,1.2\pi\cos(2\pi\times10^6 t - \frac{2\pi}{300}z + \frac{\pi}{3}) \;\left(\frac{mV}{m}\right)$$

$$\rightarrow \vec{H}(z,t) = \hat{y}\,\frac{E(z,t)}{\eta_0} = \hat{y}\,10\cos(2\pi\times10^6 t - \frac{2\pi}{300}z + \frac{\pi}{3})$$ note: $\eta_0 \simeq 120\pi\,(\Omega)$ for air.

At $t=0 \Rightarrow \vec{E}(z,0) = \hat{x}\,1.2\pi\cos(\frac{2\pi}{300}z - \frac{\pi}{3})$

$$\vec{H}(z,0) = \hat{y}\,10\cos(\frac{2\pi}{300}z - \frac{\pi}{3})$$